How Children Form Mathematical Concepts

Describing some remarkable experiments which the reader, if he has a subject handy, may perform himself. Among other things they show that in a child the historical development of geometry is reversed

by Jean Piaget

It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On the contrary, to a remarkable degree he develops them himself, independently and spontaneously. When adults try to impose mathematical concepts on a child prematurely, his learning is merely verbal; true understanding of them comes only with his mental growth.

This can easily be shown by a simple experiment. A child of five or six may readily be taught by his parents to name the numbers from 1 to 10. If 10 stones are laid in a row, he can count them correctly. But if the stones are rearranged in a more complex pattern or piled up, he no longer can count them with consistent accuracy. Although the child knows the names of the numbers, he has not yet grasped the essential idea of number: namely, that the number of objects in a group remains the same, is "conserved," no matter how they are shuffled or arranged.

On the other hand, a child of six and a half or seven often shows that he has spontaneously formed the concept of number even though he may not yet have been taught to count. Given eight red chips and eight blue chips, he will discover by one-to-one matching that the number of red is the same as the number of blue, and he will realize that the two groups remain equal in number regardless of the shape they take.

The experiment with one-to-one correspondence is very useful for investigating children's development of the number concept. Let us lay down a row of eight red chips, equally spaced about an inch apart, and ask our small subjects to take from a box of blue chips as many chips as there are on the table. Their reactions will depend on age, and we can distinguish three stages of development. A child of five or younger, on the average, will lay out blue chips to make a row exactly as long as the red row, but he will put the blue chips close together instead of spacing them. He believes the number is the same if the length of the row is the same. At the age of six, on the average, children arrive at the second stage; these children will lay a blue chip opposite each red chip and obtain the correct number. But they have not necessarily acquired the concept of number itself. If we spread the red chips, spacing out the row more loosely, the six-year-olds will think that the longer row now has more chips, though we have not changed the number. At the age of six and a half to seven, on the average, children achieve the third stage: they know that, though we close up or space out one row of chips, the number is still the same as in the other.

In a similar experiment a child is given two receptacles of identical shape and size and is asked to put beads, one at a time, into both receptacles with both hands simultaneously—a blue bead into one box with his right hand and a red bead into the other with his left hand. When he has more or less filled the two receptacles, he is asked how they compare. He is sure that both have the same number of beads. Then he is requested to pour the blue beads into a receptacle of a different size and shape. Here again we see differences in understanding according to age. The smallest children think that the number has changed: if, for instance, the beads fill the new re-

Experiment with chips demonstrates the development of the concept of number by children from the ceptacle to a higher level, they think there are more beads in it than in the original one; if to a lower level, they think there are fewer. But children near the age of seven know that the transfer has not changed the number of beads.

In short, children must grasp the principle of conservation of quantity before they can develop the concept of number. Now conservation of quantity of course is not in itself a numerical notion; rather, it is a logical concept. Thus these experiments in child psychology throw some light on the epistemology of the number concept—a subject which has been examined by many mathematicians and logicians.

The mathematicians Henri Poincaré and L. E. J. Brouwer have held that the number concept is a product of primitive intuition, preceding logical notions. The experiments just described deny this thesis, in our opinion. Bertrand Russell, on the other hand, has supported the view that number is a purely logical concept: that the idea of cardinal number derives from the logical notion of category (a number would be a category made up of equivalent categories) while the notion of ordinal number derives from the logical relationships of order. But Russell's theory does not quite fit the psychological processes as we have observed them in small children. Children at the start make no distinction between cardinal and ordinal number, and besides, the concept of cardinal number itself presupposes an order relationship. For instance, a child can build a one-to-one correspondence only if he neither forgets any of the elements nor uses the same one twice. The only way of distinguishing one unit from another is to consider it either before or after the other in time or in space, that is, in the order of enumeration.

Study of the child's discovery of spatial relationships—what may be called the child's spontaneous geometry—is no less rewarding than the investigation of his number concepts. A child's order of development in geometry seems to reverse the order of historical discovery. Scientific geometry began with the Euclidean system (concerned with figures, angles and so on), developed in the 17th century the so-called projective geometry (dealing with problems of perspective) and finally came in the 19th century to topology (describing spatial relationships in a general qualitative way—for instance, the distinction between open and closed structures, interiority and exteriority, proximity and separation). A child begins with the last: his first geometrical discoveries are topological. At the age of three he readily distinguishes between open and closed figures: if you ask him to copy a square or a triangle, he draws a closed circle; he draws a cross with two separate lines. If you show him a drawing of a large circle with a small circle inside, he is quite capable of reproducing this relationship, and he can also draw a small circle outside or attached to the edge of the large one. All this he can do before he can draw a rectangle or express the Euclidean characteristics (number of sides, angles, etc.) of a figure. Not until a considerable time after he has mastered topological relationships does he begin to develop his notions of Euclidean and projective geometry. Then he builds those simultaneously.

Curiously enough, this psychological order is much closer to modern geometry's order of deductive or axiomatic construction than the historical order of discovery was. It offers another example of the kinship between psychological construction and the logical construction of science itself.

Let us test our young subjects on projective constructions. First we set up two "fence posts" (little sticks stuck in bases of modeling clay) some 15 inches apart and ask the child to place other posts in a straight line between them. The youngest children (under the age of four) proceed to plant one post next to another, forming a more or less wavy line. Their approach is topological: the elements are joined by the simple relationship of proximity rather than by projection of a line as such. At the next stage, beyond the age of four, the child may form a straight fence if the two end posts parallel the edge of the table, or if there is some other straight line to guide him. If the end posts are diagonally across the table, he may start building the line parallel to the table's edge and then change direction and form a curve to reach the second post. Occasionally a youngster may make a straight line, but he does so only by trial-and-error and not by system.

At the age of seven years, on the average, a child can build a straight fence consistently in any direction across the table, and he will check the straightness of the line by shutting one eye and sighting along it, as a gardener lines up bean poles. Here we have the essence of the projective concept; the line is still a topological line, but the child has grasped that the projective relationship depends on the angle of vision, or point of view.

One can proceed to study this with other experiments. For instance, you stand a doll on a table and place before

age of five or younger (hands at left), *through six* (center) *to six and a half or seven* (right). *The experiment is described in detail in the text.*

Child of three draws this but not rectangle

it an object oriented in a certain direction: a pencil lying crosswise, diagonally or lengthwise with respect to the doll's line of vision, or a watch lying flat on the table or standing up. Then you ask the child to draw the doll's view of the object, or, better still, ask him to choose from two or three drawings the one that represents the doll's point of view. Not until the age of about seven or eight can a child deduce correctly the doll's angle of vision.

A similar experiment testing the same point yields the same conclusions. Objects of different shapes are placed in various positions between a light and a screen, and the child is asked to predict the shape of the shadow the object will cast on the screen.

Ability to coordinate different perspectives does not come until the age of 9 or 10. This is illustrated by an experiment I suggested some time ago to my collaborator Dr. Edith Meyer. The experimenter sits at a table opposite the child, and between the child and herself she places a cardboard range of mountains. The two see the range from opposite perspectives. The child is then asked to select from several drawings the ones that picture both his own and the opposite person's views of the mountain range. Naturally the youngest children can pick out only the picture that corresponds to their own view; they imagine that all the points of view are like their own. What is more interesting, if the child changes places with the experimenter and sees the mountains from the other side, he now thinks that his new view is the only correct one; he cannot reconstruct the point of view that was his own just a little while before. This is a clear example of the egocentricity so characteristic of children—the primitive reasoning which prevents them from understanding that there may be more than one point of view.

It takes a considerable evolution for children to come, at around the age of 9 or 10, to the ability to distinguish between and coordinate the different possible perspectives. At this stage they can grasp projective space in its concrete or practical form, but naturally not in its theoretical aspects.

At the same time the child forms the concept of projective space, he also constructs Euclidean space; the two kinds of construction are based upon one another. For example, in lining up a straight row of fence posts he may not only use the sighting method but may line up his hands parallel to each other to give him the direction. That is, he is applying the concept of conservation of direction, which is a Euclidean principle. Here is another illustration of the fact that children form mathematical notions on a qualitative or logical basis.

The conservation principle arises in various forms. There is first the conservation of length. If you place a block on another of the same length and then push one block so that its end projects beyond the other, a child under six will suppose that the two blocks are no longer of equal length. Not until near the age of seven, on the average, does the child understand that what is gained at one end of the block is lost at the other. He arrives at this concept of the conservation of length, be it noted, by a process of logic.

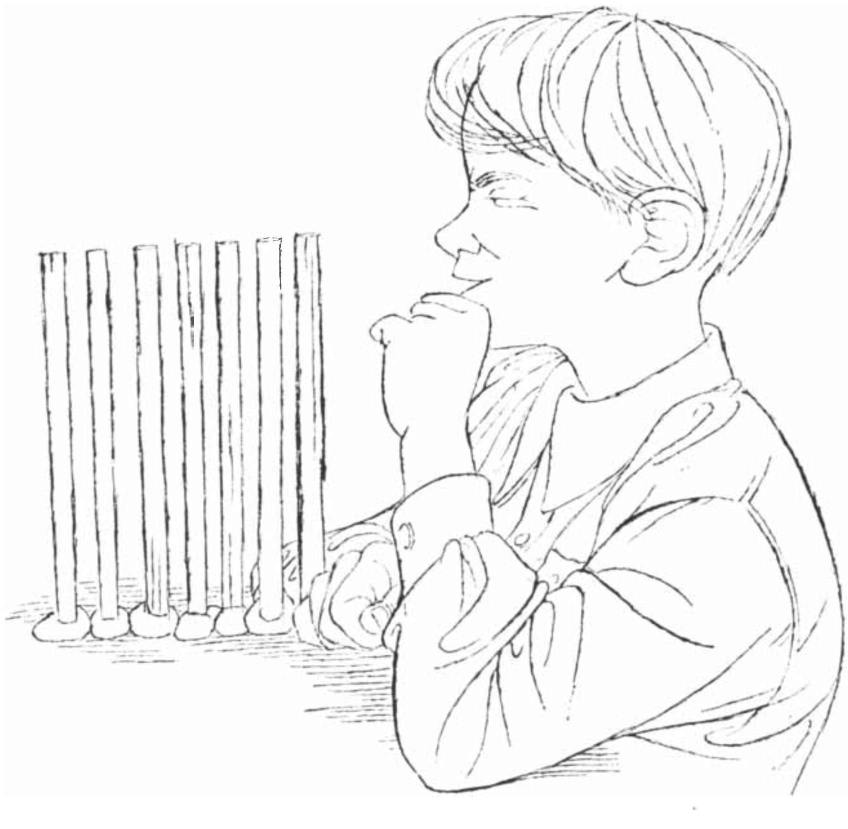
Experiments on a child's discovery of the conservation of distance are especially illuminating. Between two small toy trees standing apart from each other on a table you place a wall formed of a block or a thick piece of cardboard, and you ask the child (in his own language, of course) whether the trees are still the same distance apart. The smallest children think the distance has changed; they are simply unable to add up two parts of a distance to a total distance. Children of five or six believe the distance has been reduced, claiming that the width of the wall does not count as distance; in other words, a filled-up space does not have the same value as an empty space. Only near the age of seven do children come to the realization that intervening objects do not change the distance.

However you test them, you find the same thing true: children do not appreciate the principle of conservation of length or surface until, somewhere around the age of seven, they discover the reversibility that shows the original quantity has remained the same (*e.g.*, the realignment of equal-length blocks, the removal of the wall, and so on). Thus the discovery of logical relationships is a prerequisite to the construction of geometrical concepts, as it is in the formation of the concept of number.

This applies to measurement itself, which is only a derived concept. It is interesting to study how children spontaneously learn to measure. One of my collaborators, Dr. Inhelder, and I have made the following experiment: We show the child a tower of blocks on a table and ask him to build a second tower of the same height on another table (lower or higher than the first) with blocks of a different size. Naturally we provide the child with all the necessary measuring tools. Children's attempts to deal with this problem go through a fascinating evolution. The youngest children build up the second tower to the same visual level as the first, without worrying about the difference in height of the tables. They compare the towers by stepping back and sighting them. At a slightly more advanced stage a child lays a long rod across the tops of the two towers to make sure that they are level. Somewhat later he notices that the base of his tower is not at the same level as the model's. He then wants to place his tower next to the model on the same table to compare them. Reminded that the rules of the game forbid him to move his tower, he begins to look around for a measuring standard. Interestingly enough, the first that comes to his mind is his own body. He puts one hand on top of his tower and the other at its base, and then, trying to keep his hands the same distance apart, he moves over to the other tower to compare it. Children of about the age of six often carry out this work in a most assured manner, as if their hands could not change position on the way! Soon they discover that the method is not reliable, and then they resort to reference points on the body. The child will line up his shoulder with the top of his tower, mark the spot opposite the base on his thigh with his hand and walk over to the model to see whether the distance is the same.

Eventually the idea of an independent measuring tool occurs to the child. His first attempt in this direction is likely to be the building of a third tower next to and the same height as the one he has already erected. Having built it, he moves it over to the first table and matches it against the model; this is allowed by the rules. The child's arrival at this stage presupposes a process of logical reasoning. If we call the model tower A, the second tower C and the movable tower B, the child has reasoned that $B=C$ and $B=A$, therefore $A=C$.

Later the child replaces the third tower with a rod, but at first the rod must



Child of seven straightens a row of "fence posts" by sighting along them

Child of six measures the height of a tower of blocks with her body

be just the same length as the height of the tower to be measured. He then conceives the idea of using a longer rod and marking the tower height on it with his finger. Finally, and this is the beginning of true measurement, he realizes that he can use a shorter rod and measure the height of the tower by applying the rod a certain number of times up the side.

The last discovery involves two new operations of logic. The first is the process of division which permits the child to conceive that the whole is composed of a number of parts added together. The second is the displacement, or substitution, which enables him to apply one part upon others and thus to build a system of units. One may therefore say that measurement is a synthesis of division into parts and of substitution, just as number is a synthesis of the inclusion of categories and of serial order. But measurement develops later than the number concept, because it is more difficult to divide a continuous whole into interchangeable units than to enumerate elements which are already separate.

To study measurement in two dimensions, we give the child a large sheet of paper with a pencil dot on it and ask him to put a dot in the same position on another sheet of the same size. He may use rods, strips of paper, strings, rulers or any other measuring tools he needs. The youngest subjects are satisfied to make a visual approximation, using no tools. Later a child applies a measuring tool, but he measures only the distance of the point from the side or bottom edge of the paper and is surprised that this single measurement does not give him the correct position. Then he measures the distance of the point from a corner of the paper, trying to keep the same slant (angle) when he applies the ruler to his own sheet. Finally, at about the age of eight or nine, he discovers that he must break up the measurement into two operations: the horizontal distance from a side edge and the perpendicular distance from the bottom or top edge. Similar experiments with a bead in a box show that a child discovers how to make three-dimensional measurements at about the same age.

Measurement in two or three dimensions brings us to the central idea of Euclidean space, namely the axes of coordinates—a system founded on the horizontality or verticality of physical objects. It may seem that even a baby should grasp these concepts, for after all it can distinguish between the upright and lying-down positions. But actually the representation of vertical and horizontal lines brings up quite another problem from this subjective awareness of postural space. Dr. Inhelder and I have studied it with the following experiments: Using a jar half-filled with colored water, we ask our young subjects to predict what level the water will take when the jar is tipped one way or another. Not until the age of nine, on the average, does a child grasp the idea of horizontality and predict correctly. Similar experiments with a plumb line or a toy sailboat with a tall mast demonstrate that comprehension of verticality comes at about the same time. The child's tardiness in acquiring these concepts is not really surprising, for they require not only a grasp of the internal relationships of an object but also reference to external elements (*e.g.*, a table or the floor or walls of the room).

When a child has discovered how to construct these coordinate axes by reference to natural objects, which he does at about the same time that he conceives the coordination of perspectives, he has completed his conception of how to represent space. By that time he has developed his fundamental mathematical concepts, which spring spontaneously from his own logical operations.

The experiments I have described, simple as they are, have been surprisingly fruitful and have brought to light many unexpected facts. These facts are illuminating from the psychological and pedagogical points of view; more than that, they teach us a number of lessons about human knowledge in general.